Math 5B Test 3
SAMPLE
(11.8-11.11, 8.1,10.1, 10.3)

100 POINTS **NAME:** Solutions

No scratch paper. Show all work clearly on test paper. No credit will be given for solutions if work is not shown. Only non-graphing calculators are allowed. Unless otherwise specified, the answer to series questions should be given using sigma notation. Unless otherwise stated, you do not need to find the radius of convergence.

(1) FIND THE INTERVAL OF CONVERGENCE FOR EACH OF THE FOLLOWING.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x+3)^n}{\sqrt{n}}$

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{2^n(x+3)^n}\right| = \lim_{n\to\infty}\underbrace{\frac{2\sqrt{n}}{\sqrt{n+1}}}_{2}|x+3|$$

$$L = 2|x+3|$$

If $L<1$, series is absolutely convergent

If $L>1$, series diverges.

If $L=1$, ratio test inconclusive

so check endpoints sepately

$2|x+3|<1$

$\Rightarrow |x+3|<\frac{1}{2}$

$-\frac{1}{2}<x+3<\frac{1}{2}$

$-\frac{7}{2}<x<-\frac{5}{2}$

$-\frac{7}{2}$ $\quad -3 \quad$ $-5/2$ $\quad R=1/2$

$x=-\frac{7}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x+3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{2^n(-\frac{1}{2})^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ conv

$x=-\frac{5}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x+3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{2^n(1/2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ div

$\boxed{[-7/2, -5/2)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2\bullet 4\bullet 6\bullet\cdots\bullet(2n)}$

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2x^{n+1}}{\cancel{2\cdot4\cdot6\cdots(2n)}\cdot(2(n+1))}\cdot\frac{\cancel{2\cdot4\cdot6\cdots(2n)}}{n^2x^n}\right|$$

$$=\lim_{n\to\infty}|x|\frac{(n+1)^2}{n^2(2n+2)} = 0 \text{ for all } x$$

$\boxed{(-\infty,\infty)}$

Additionally, find a_7, the seventh term of the series (no need to simplify).

$$\boxed{\frac{7^2x^7}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14}}$$

when $n=7$, $2n=14$ so last factor in denominator is 14.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}(x-5)^n$

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-5)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{n!\,(x-5)^n}\right| = \lim_{n\to\infty}\frac{n+1}{3}|x-5| = \infty$$

unless $x=5$.

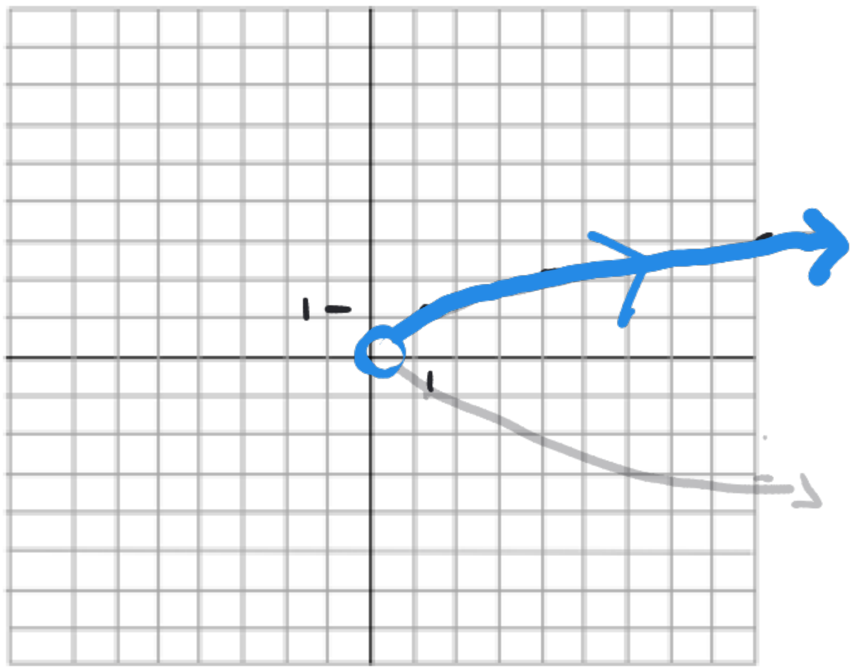
$\boxed{\text{Converges at } x=5 \text{ only}}$

(2) Eliminate the parameter and sketch the curve, showing direction of increasing t . $\begin{cases} x = e^{2t} = (e^t)^2 > 0 \\ y = e^t > 0 \end{cases}$

Eliminate t

$x = y^2$; $x > 0$, $y > 0$

Just the blue part.



(3) Find the Maclaurin series for f(x)=cos2x directly, using the definition.

↳ otherwise, the easier way would be to substitute 2x for x into $\cos x = \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!}$

	at x	at x=a=0
f	$\cos 2x$	1
f '	$-2\sin 2x$	0
f ''	$-4\cos 2x$	-4
f '''	$8\sin 2x$	0
	$16\cos 2x$	16
f^n	$f^n(x)=$	$f^n(0)=$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

$$= 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 - \frac{64}{6!}x^6 + \cdots$$

now see pattern.

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{(2n)}}{(2n)!} x^{2n}$$

Cannot find general term easily, so go ahead and write out some terms

(4) Find the Maclaurin series for $x^4 e^{x^3}$

(There are easy ways and there are hard ways this can be done)

↳ generating directly from the defintion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty, \infty)$$

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

$$x^4 e^{x^3} = x^4 \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{3n+4}}{n!}}$$

(5) Find the Taylor series for $f(x)=1/x^2$ centered at a=2. (Assume that f has a power series expansion.

	at x	at x=a=2
f	x^{-2}	
f '	$-2x^{-3}$	
f ''	$3 \cdot 2x^{-4}$	
f '''	$-4 \cdot 3 \cdot 2x^{-5}$	
f^n	$f^n(x) = \frac{(-1)^n (n+1)!}{x^{n+2}}$	$f^n(2) = \frac{(-1)^n (n+1)!}{2^{n+2}}$

$$\frac{1}{x^2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{2^{n+2}\, n!}(x-2)^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^{n+2}}(x-2)^n}$$

(6) Find the length of the curve $y=x^{2/3}$ from (1,1) to $(2\sqrt{2},2)$ Can do either dx or dy

dx $\quad y=x^{2/3} \quad 0\le x\le 2^{3/2}$ OR dy $\quad x=y^{3/2} \quad 1\le y\le 2$

$$L=\int_1^{2^{3/2}}\sqrt{1+\left(\tfrac{2}{3}x^{-1/3}\right)^2}\,dx=\int_1^{2^{3/2}}\sqrt{1+\frac{4}{9x^{2/3}}}\,dx$$

$$=\int_1^{2^{3/2}}\frac{1}{3x^{1/3}}\sqrt{9x^{2/3}+4}\,dx \qquad u=9x^{2/3}+4,\quad du=6x^{-1/3}dx$$

$$=\frac{1}{18}\int_1^{22}u^{1/2}\,du=\frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big]_{13}^{22}$$

$$\boxed{\frac{1}{27}\left(22^{3/2}-13^{3/2}\right)}$$

$$L=\int_1^2\sqrt{1+\left(\tfrac{3}{2}x^{1/2}\right)^2}\,dx$$

$$=\int_1^2\sqrt{1+\tfrac{9}{4}x}\,dx \qquad u=1+\tfrac{9}{4}x,\quad du=\tfrac{9}{4}dx$$

$$\frac{4}{9}\int_{\frac{13}{4}}^{\frac{11}{2}}u^{1/2}=\frac{4}{9}\cdot\frac{2}{3}u^{3/2}\Big]_{\frac{13}{4}}^{\frac{11}{2}}$$

$$=\frac{8}{27}\left(\left(\frac{11}{2}\right)^{3/2}-\left(\frac{13}{4}\right)^{3/2}\right)$$

$$=\frac{8}{27}\left(\left(\frac{22}{4}\right)^{3/2}-\left(\frac{13}{4}\right)^{3/2}\right)$$

$$\frac{1}{27}\left(22^{3/2}-13^{3/2}\right)$$

same

(7) (

24. Match the parametric equations with the graphs labeled I–VI. Give reasons for your choices. (Do not use a graphing device.) There are many ways to approach

IV (a) $x=t^3-2t,\quad y=t^2-t$

VI (b) $x=t^3-1,\quad y=2-t^2$

V (c) $x=\sin 3t,\quad y=\sin 4t$

III (d) $x=t+\sin 2t,\quad y=t+\sin 3t$

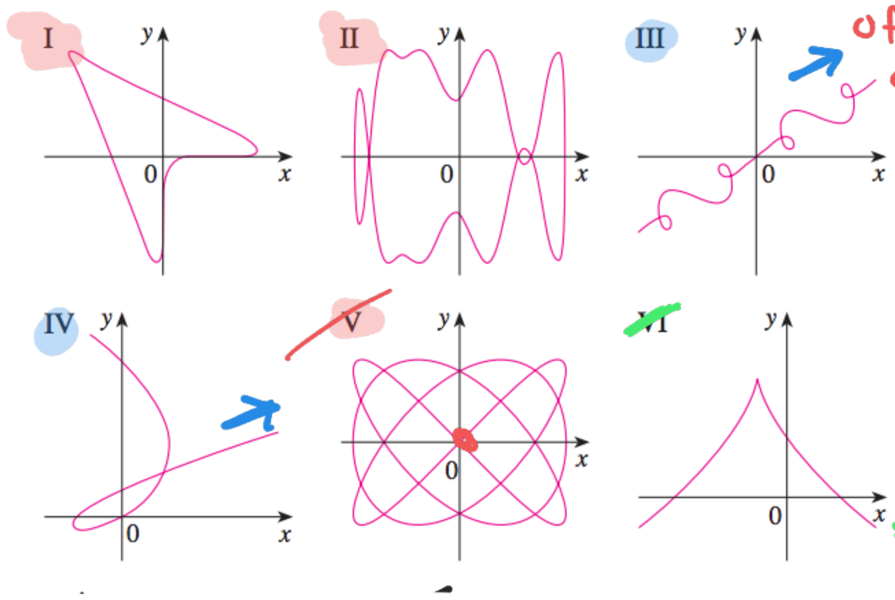
I (e) $x=\sin(t+\sin t),\quad y=\cos(t+\cos t)$

II (f) $x=\cos t,\quad y=\sin(t+\sin 5t)$

As $t\to\infty$, x & $y\to\infty$ $\Rightarrow$ III and IV (III oscillates ↗)

Only one where $y\to-\infty$ as $t\to\infty$

Comparing the 3 left all have $-1\le x\le 1$, $-1\le y\le 1$



of the 3 graphs left only V goes through $(0,0)\Rightarrow$ (c)

Down to equation e, f with graphs I, II.

For (f) $y=\sin(t+\sin 5t)$ varies much more since period of $\sin 5t=\frac{2\pi}{5}$

y varies more in graph II

(8) Using the geometric series for $\frac{1}{1-x}$ find a power series representation for $\frac{5x}{1+3x}$ and determine the radius of convergence.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x|<1$$

substitute $-3x$ for x

$$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3x)^n \quad |-3x|<1$$

$$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-1)^n 3^n x^n \quad |x|<1/3$$

$R = \frac{1}{3}$

$$\boxed{\frac{5x}{1+3x} = \sum_{n=0}^{\infty} 5(-1)^n 3^n x^{n+1} \quad |x|<1/3}$$

(9) Use series to compute $\int_0^{1/2} x^2 e^{-x^2}\,dx$ with $|\text{error}| < 0.001$.

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty,\infty)$$

$$e^{-x^2} = 1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$x^2e^{-x^2} = x^2-x^4+\frac{x^6}{2!}-\frac{x^8}{3!}+\cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}$$

$$\int_0^{1/2} x^2e^{-x^2}\,dx = \left.\frac{1}{3}x^3-\frac{1}{5}x^5+\frac{1}{14}x^7-\frac{1}{54}x^9+\cdots\right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!(2n+3)}$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)^3-\frac{1}{5}\left(\frac{1}{2}\right)^5+\frac{1}{14}\left(\frac{1}{2}\right)^7-\frac{1}{54}\left(\frac{1}{2}\right)^9+\cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+3}\, n!(2n+3)}$$

$$= \frac{1}{24}-\frac{1}{160}+\frac{1}{1792}-$$

Alternating series.
First term < .001

$$\approx \frac{1}{24}-\frac{1}{160} = \boxed{.035}$$

(10) (a) Approximate the function $f(x) = x\ln x$ by $T_3(x)$, the third degree Taylor Polynomial centered at a=1.
(b) Use Taylor's Inequality to estimate the accuracy of the approximation when x lies in the interval $0.9 \le x \le 1.1$
(c) Use $T_3(x)$ to approximate (1.01) ln(1.01)

	at x	at x=a=1
f	$x\ln x$	0
f '	$\ln x + 1$	1
f ''	$\frac{1}{x}$	1
f '''	$-\frac{1}{x^2}$	-1
	$\frac{2}{x^3}$	
f^n	$f^n(x)=$	$f^n(1)=$

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$\boxed{T_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3}$$

Find M - upper bound for $f^{(4)}(x)$ on $0.9 \le x \le 1$

$$f^{(4)}(x) < M \quad 0.9 \le x \le 1.1$$

$$\frac{2}{x^3} < M = \frac{2}{(0.9)^3}$$

$$|R_3(x)| \le \frac{M}{4!}(x-1)^4$$

Largest value of $x-1$ is $1.1 - 1 = 0.1$

$$|R_3(x)| \le \frac{\frac{2}{(0.9)^3}}{4!}(.1)^4 = \frac{2(.1)^4}{24(0.9)^3}$$

$$\approx \boxed{0.00001}$$

$x - 1 = 1.01 - 1 = .01$

$$1.01 \ln 1.01 \approx T_3(1.01)$$

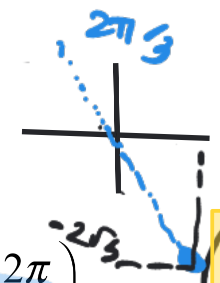
$$= .01 + \frac{1}{2}(.01)^2 - \frac{1}{6}(.01)^3$$

$$\approx \boxed{.01004983\overline{3}}$$

(compare to calculator value - should be within 0.00001

(11)

(a) Convert from polar to rectangular coordinates $\left(-4, \frac{2\pi}{3}\right)$: $\boxed{(2, -2\sqrt{3})}$



think about your answer - reasonable?

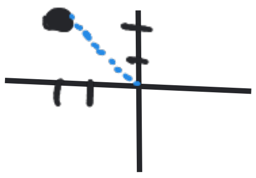
$x = r\cos\theta = -4\cos\frac{2\pi}{3} = -4(-1/2) = 2$

$y = r\sin\theta = -4\sin\frac{2\pi}{3} = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$

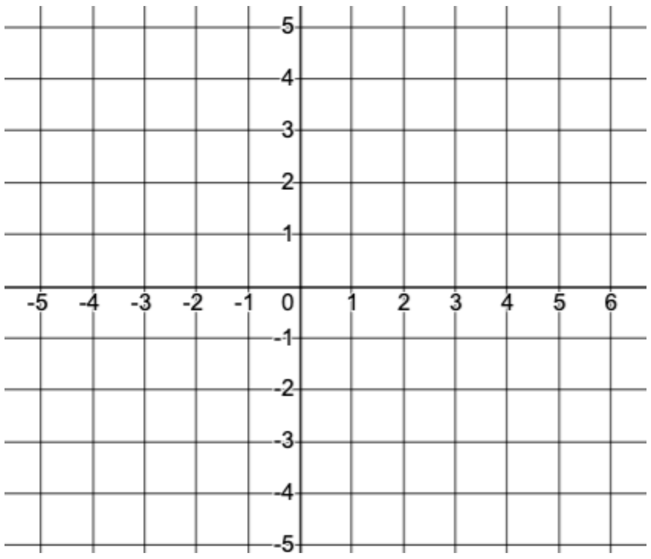
(b) Convert from rectangular to polar coordinates $(-2,2)$: $\boxed{\left(2\sqrt{2}, \frac{3\pi}{4}\right)}$

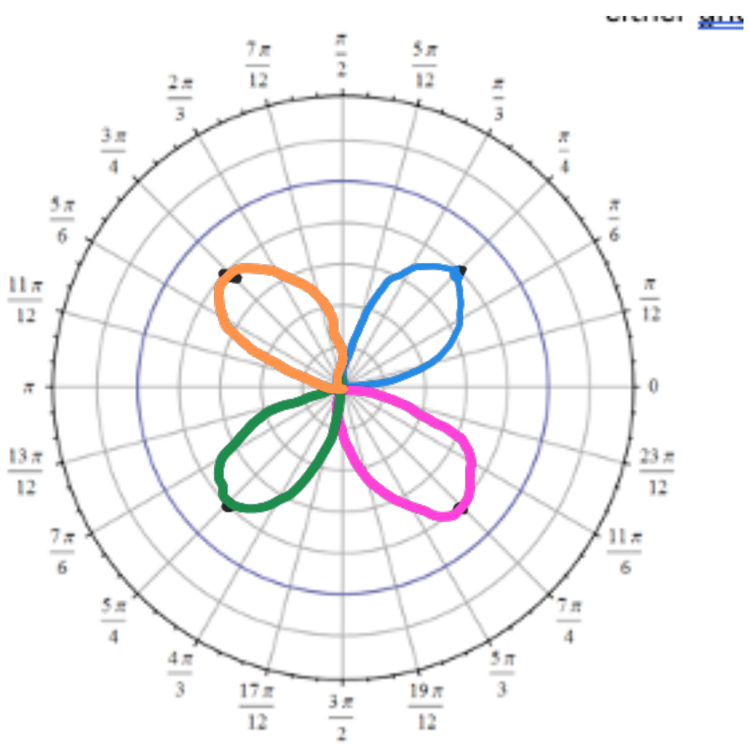
$r^2 = x^2+y^2 = 4+4=8$ $\quad r=\sqrt{8} = 2\sqrt{2}$

$\tan\theta = \frac{2}{-2} = -1$, QII $\Rightarrow \theta = \frac{3\pi}{4}$



(c) Graph the polar function: $r = 3\sin 2\theta$ (You can use either grid)





Rose with $2n=2(2)=4$ leaves

Tip where $r=3 \Rightarrow \sin 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

spacing $\frac{2\pi}{4} = \frac{\pi}{2}$